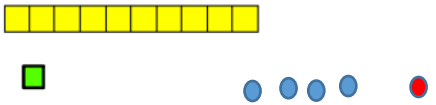
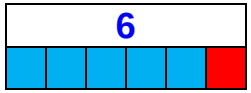
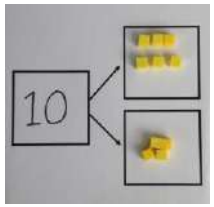
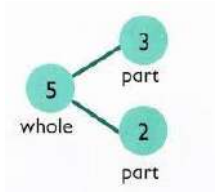
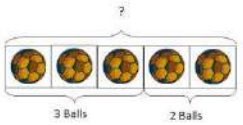
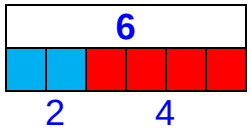
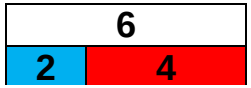
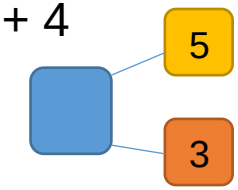
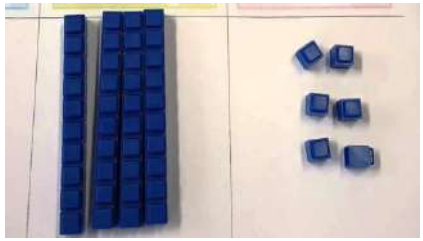
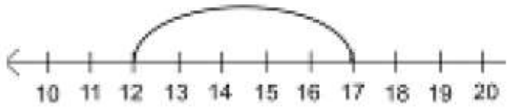
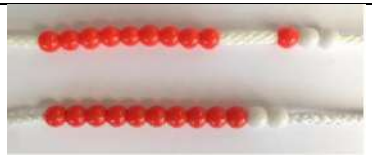
Progression in Calculations

Addition

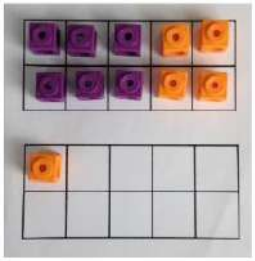
Objective and Strategies	Concrete	Pictorial	Abstract
Adding 1 more than…	  Use a playground number line	Use pictures of dienes or sticks and dots   5 + 1 = 6 Use bar model	5 + 1 = 6  + 1 = 6 6 =  + 1
Combining two parts to make a whole: part-whole model	   Use cubes to add two numbers together as a group or in a bar. 	    Use pictures to add two numbers together as a group or in a bar.  Use a bar model  2 4 	4 + 3 = 7 10= 6 + 4  Use the part-part whole diagram as shown above to move into the abstract.

Starting at the bigger number and counting on	 Start with the larger number on the bead string and then count on to the smaller number 1 by 1 to find the answer. Use a playground number line or one drawn in chalk by the children.  Use of dienes  Use of a counting stick 	12 + 5 = 17  Start at the larger number on the number line and count on in ones or in one jump to find the answer. Use of a counting stick diagram 	5 + 12 = 17 Place the larger number in your head and count on the smaller number to find your answer.

Regrouping to make 10.

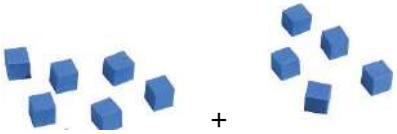


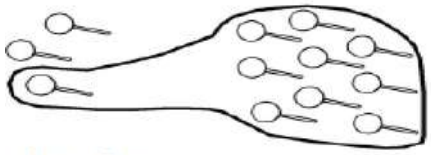
6 + 5 = 11



Start with the bigger number and use the smaller number to make 10.

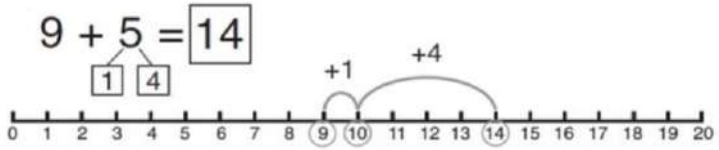
Use of dienes, adding with single cubes, until ten ones can be exchanged for one ten stick (reinforcement of place value too!)



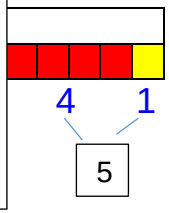


Use pictures or a number line. Regroup or partition the smaller number to make 10.

3 + 9 =



6 + 5 = 11



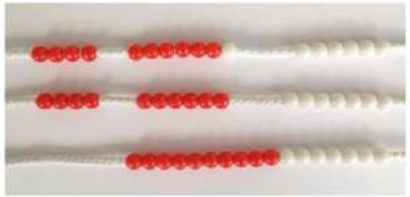
Use of bar models

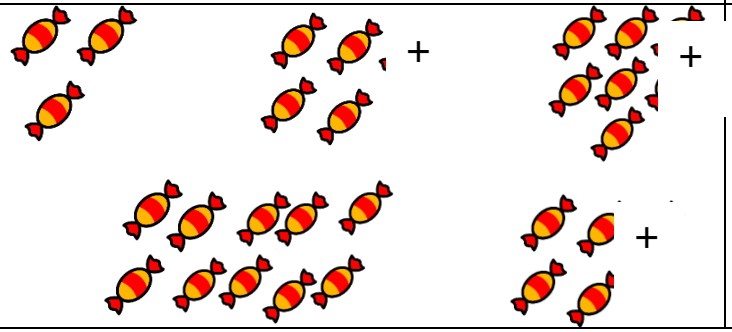
7 + 4= 11

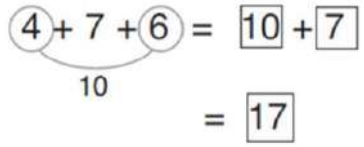
If I am at seven, how many more do I need to make 10. How many more do I add on now?

Adding three single digits

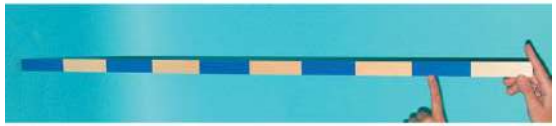
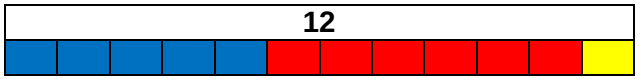
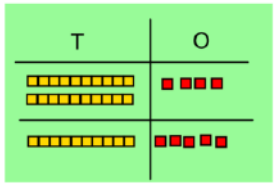
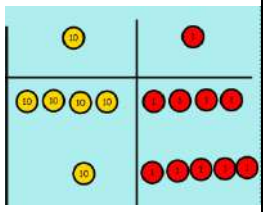
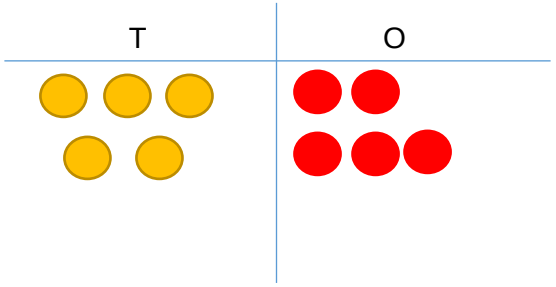
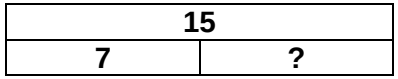
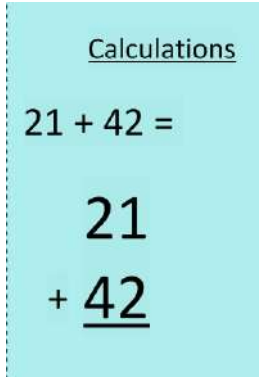
4 + 7 + 6= 17
Put 4 and 6 together to make 10. Add on 7.

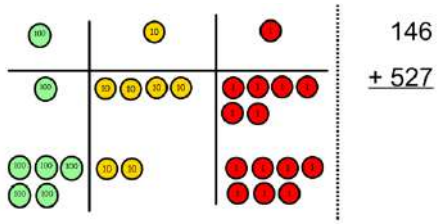
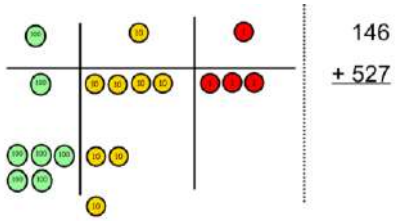
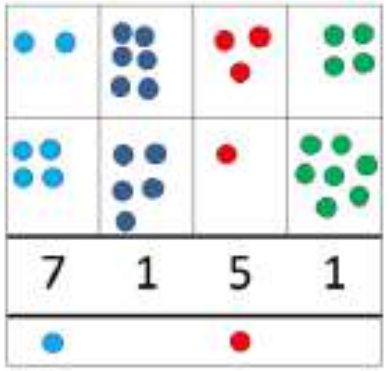
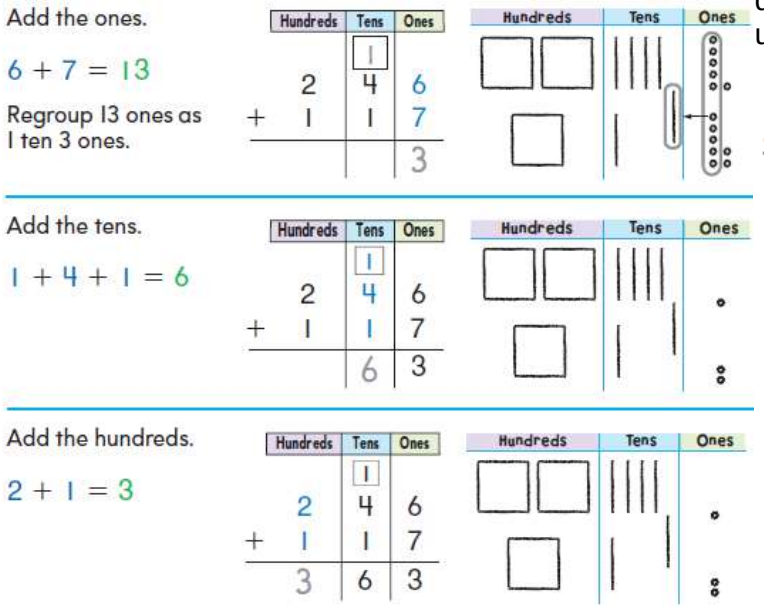






Combine the two numbers that make 10 and then add on the remainder.

	Following on from making 10, make 10 with 2 of the digits (if possible) then add on the third digit. Use dienes  Use a counting stick as a number line 	Add together three groups of objects. Draw a picture to recombine the groups to make 10. Use of a bar model to show part, part, whole.  	
Column method- no regrouping	24 + 15= Add together the ones first then add the tens. Use the Base 10 blocks first before moving onto place value counters.  	After practically using the base 10 blocks and place value counters, children can draw the counters to help them to solve additions.  Use of sticks and dots  Use of bar models for missing boxes and number lines to check.  	

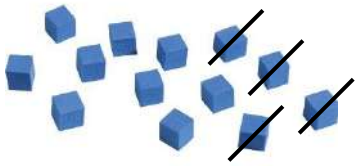
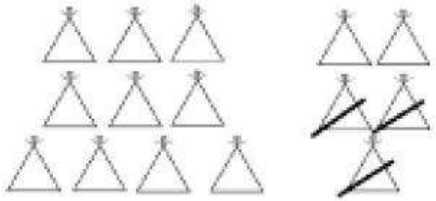
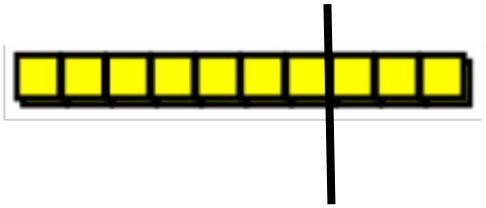
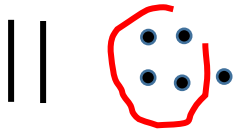
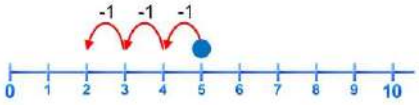
Column method- regrouping	Make both numbers on a place value grid.  146 + 527 Add up the units and exchange 10 ones for one 10.  146 + 527 Add up the rest of the columns, exchanging the 10 counters from one column for the next place value column until every column has been added. This can also be done with Base 10 to help children clearly see that 10 ones equal 1 ten and 10 tens equal 100.	Children can draw a pictoral representation of the columns and place value counters to further support their learning and understanding.  7 1 5 1 Use of sticks and dots  Add the ones. 6 + 7 = 13 Regroup 13 ones as 1 ten 3 ones. Add the tens. 1 + 4 + 1 = 6 Add the hundreds. 2 + 1 = 3	Start by partitioning the numbers before moving on to clearly show the exchange below the addition. 20 + 5 40 + 8 60 + 13 = 73 As the children move on, introduce decimals with the same number of decimal places and different. Money can be used here. 536 + 85 621 1 1 72.8 + 54.6 127.4 1 1 £23.59 + £7.55 £31.14 23.361 9.080 59.770 + 1.300 93.511 2 1 2

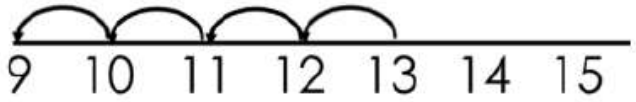
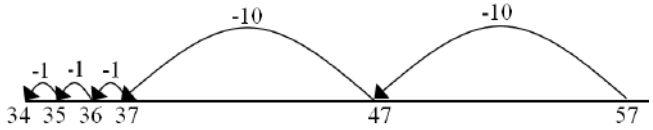
	 As children move on to decimals, money and decimal place value counters can be used to support learning.		

Notes

- **Bar models can be used for missing box problems with most of the objectives**
- **Number lines can be used for time problems, involving addition and subtraction**

Subtraction

Objective and Strategies	Concrete	Pictorial	Abstract
Taking away ones	Use physical objects, counters, cubes etc to show how objects can be taken away.  6 – 2 = 4  Use dienes  Use of number lines on the playground or number lines drawn by the children (and use of a dice)  Use of counting sticks 	Cross out drawn objects to show what has been taken away.  15 – 3 = 12 Draw dienes representations  **10 - 3 = 7** Use of sticks and dots  **25 - 4 = 21** Use of a number line 	18 -3= 15 8 – 2 = 6 Missing box ideas 8 - ☐ = 6 and reversal of the algorithm 6 = 8 - ☐

Counting back	Make the larger number in your subtraction. Move the beads along your bead string as you count backwards in ones.  13 – 4 Use counters and move them away from the group as you take them away counting backwards as you go.  Use of dienes  6 - 1 = 5  10 - 5 = 5	Count back on a number line or number track  Start at the bigger number and count back the smaller number showing the jumps on the number line.  This can progress all the way to counting back using two 2 digit numbers.	Put 13 in your head, count back 4. What number are you at? Use your fingers to help. 13 - 2 = 11 11 = 13 - 2 ‘What is 2 less than 13?’

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Find the difference

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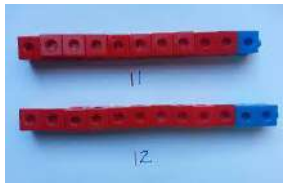
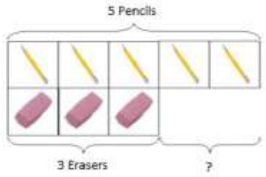
Compare amounts and objects to find the difference.



Use cubes or dienes to build towers or make bars to find the difference



Use basic bar models with items to find the difference

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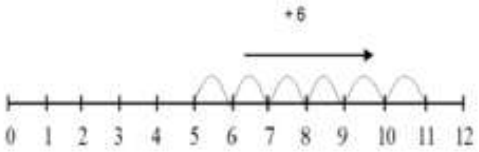
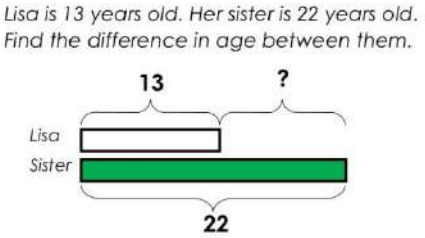


Use of number lines

Draw bars to find the difference between 2 numbers.

Comparison bar models



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Hannah has 23 sandwiches, Helen has 15 sandwiches. Find the difference between the number of sandwiches.

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Part Part Whole Model

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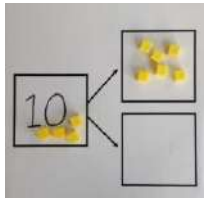

Link to addition- use the part whole model to help explain the inverse between addition and subtraction.

If 10 is the whole and 6 is one of the parts. What is the other part?

10 - 6 =

Use of dienes 9 - 5 = 4

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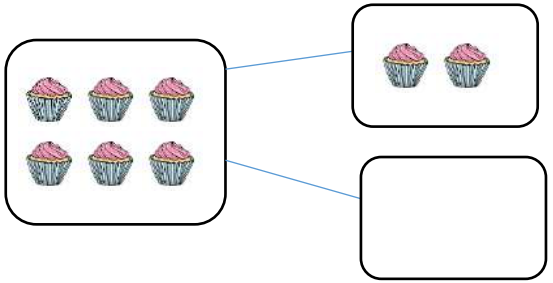
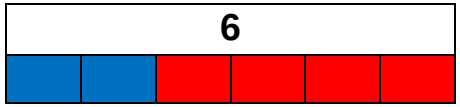
Use a pictorial representation of objects to show the part part whole model.



Use of bar models



2 4

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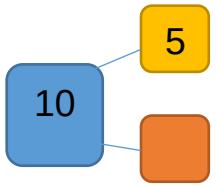

Move to using numbers within the part whole model.

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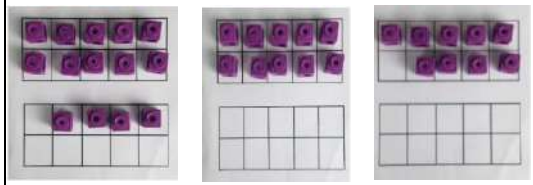
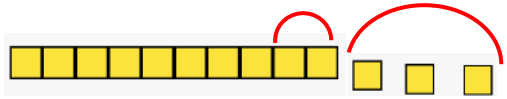
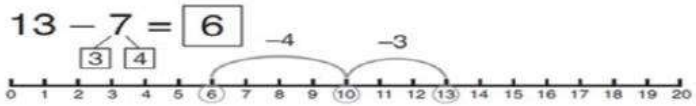
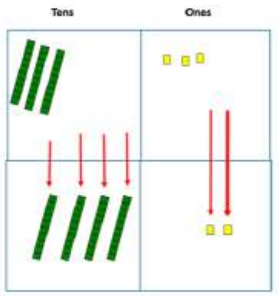
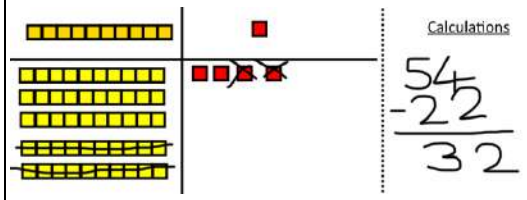
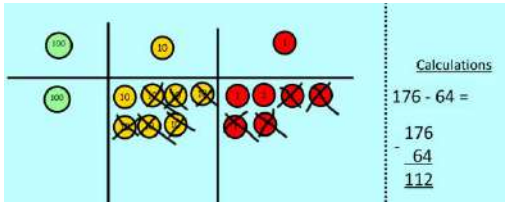
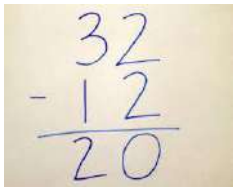
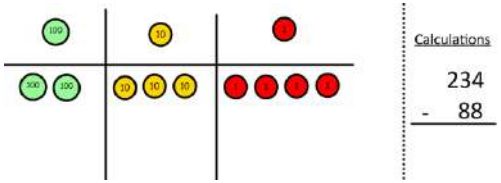
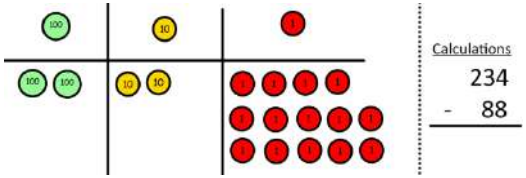
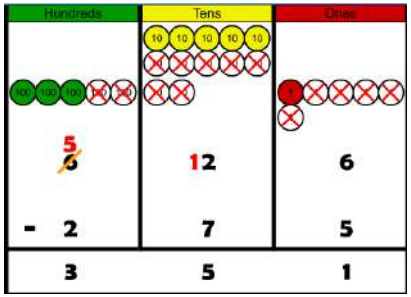
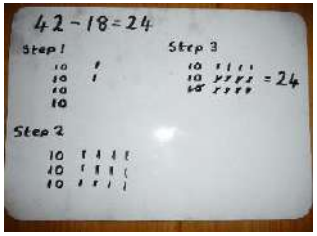
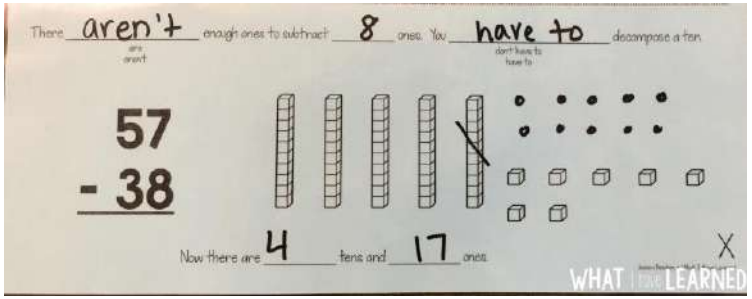
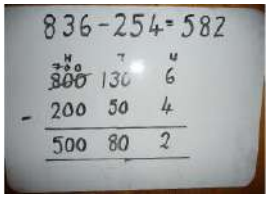
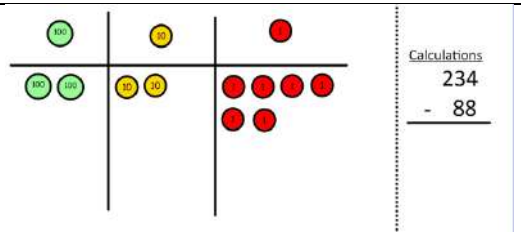
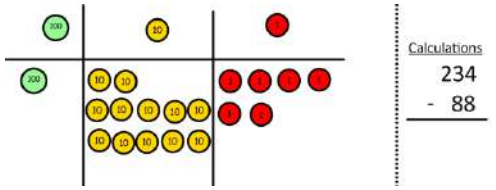
</table>

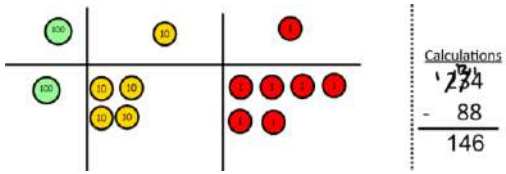
Make 10	14 – 9 =  Make 14 on the ten frame. Take away the four first to make 10 and then takeaway one more so you have taken away 5. You are left with the answer of 9. Use of dienes 13 - 5 =  - 2 - 3 13 - 3 - 2 =	Use of number lines  Start at 13. Take away 3 to reach 10. Then take away the remaining 4 so you have taken away 7 altogether. You have reached your answer.	16 – 8= How many do we take off to reach the next 10? How many do we have left to take off?
Column method without regrouping	 Use Base 10 to make the bigger number then take the smaller number away. Show how  	  Draw the Base 10 or place value counters alongside the written calculation to help to show working.	47 – 24 = 23 40 + 7 – 20 + 4 20 + 3

	you partition numbers to subtract. Again make the larger number first.	Sticks and dots subtraction and use of number line to check answers. 55 - 23 = 	This will lead to a clear written column subtraction. 
Column method with regrouping	**Use Base 10** to start with before moving on to place value counters. Start with one exchange before moving onto subtractions with 2 exchanges. Make the larger number with the place value counters  Start with the ones, can I take away 8 from 4 easily? I need to exchange one of my tens for ten ones.  Now I can subtract my ones. Now look at the tens, can I take away 8 tens easily? I need to exchange one hundred for ten tens.	 Draw the counters onto a place value grid and show what you have taken away by crossing the counters out as well as clearly showing the exchanges you make.  When confident, children can find their own way to record the exchange/regrouping. Just writing the numbers as shown here shows that the child understands the method and knows when to exchange/regroup. Sticks and dots 	 Children can start their formal written method by partitioning the number into clear place value columns.  Moving forward the children use a more compact method. This will lead to an understanding of subtracting any number including decimals.





Now I can take away eight tens and complete my subtraction



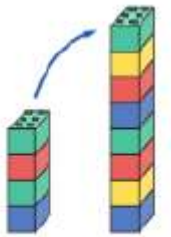
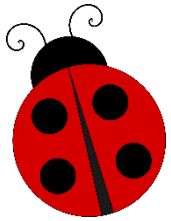
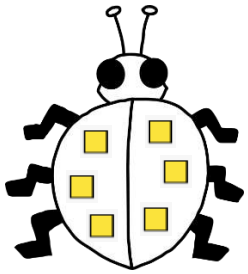
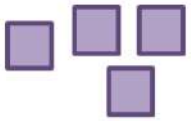
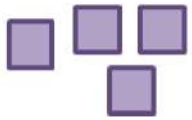
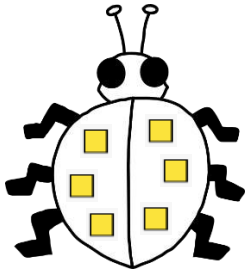
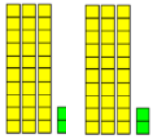
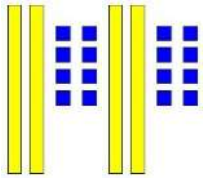
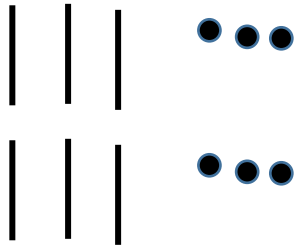
Show children how the concrete method links to the written method alongside your working. Cross out the numbers when exchanging and show where we write our new amount.

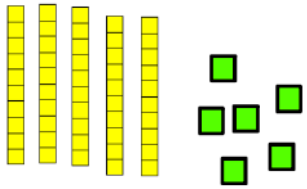
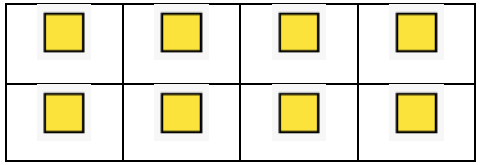
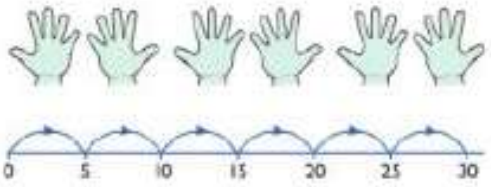
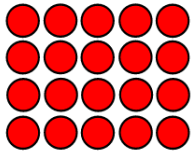
$$\begin{array}{r} \ 5\ \ 12\ \ \ 1 \\ 2\ \not{6}\ \not{3}\ .\ \mathbf{0} \\ -\quad 2\ 6\ .\ 5 \\ \hline 2\ 3\ 6\ .\ 5 \end{array}$$

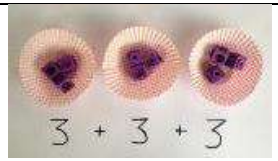
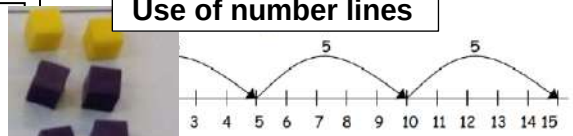
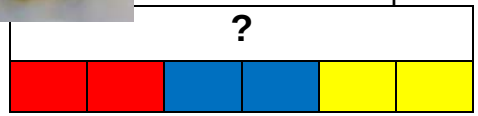
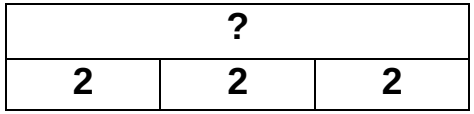
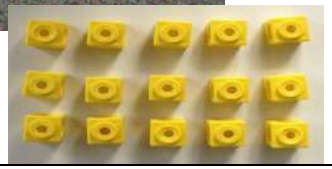
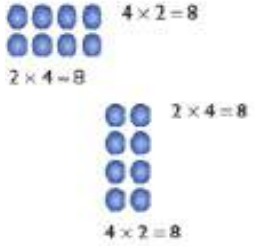
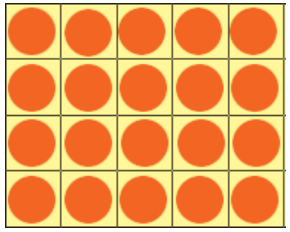
Notes

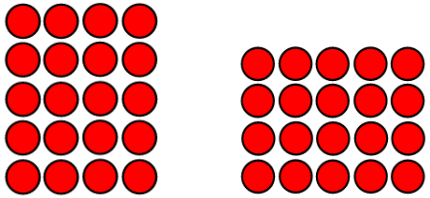
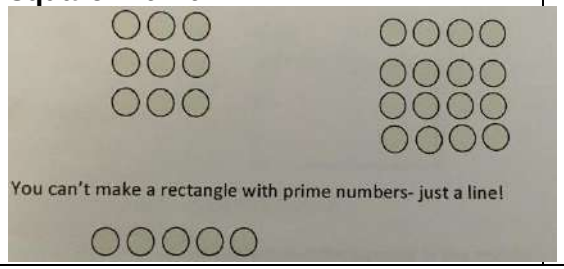
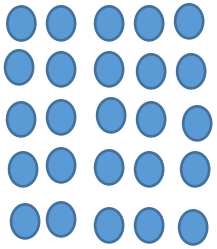
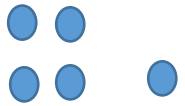
- **Bar models for missing box problems**
- **Number lines for time problems**

Multiplication

Objective and Strategies	Concrete	Pictorial	Abstract
Doubling	Use practical activities to show how to double a number.  double 4 is 8 4 × 2 = 8 Use of dienes  or 	Draw pictures to show how to double a number. Double 4 is 8   Diagrams to add dots or sticks to.  or 	16 10 6 x2 x2 20 12 Partition a number and then double each part before recombining it back together. 3 + 3 = 6
Doubling with double-digit numbers (KS2)	Use dienes a.  with no exchanging b.  with exchanging of	Use of sticks and dots to help with exchange process. 	What is double 23? What is double 47? What is double 87?

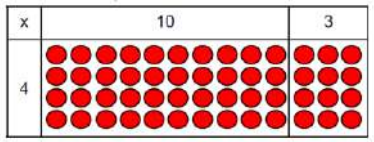
	the ones c.  with exchanging of tens and ones		
Counting in multiples	  Count in multiples supported by concrete objects in equal groups. Use of dienes on a mat 	 Use a number line or pictures to continue support in counting in multiples.  5, 10, 15, 20	Count in multiples of a number aloud. Write sequences with multiples of numbers. 2, 4, 6, 8, 10 5, 10, 15, 20, 25 , 30 ‘I have 12 cubes and each row has 4 cubes – how many rows are there?’

Repeated addition	   Use different objects to add equal groups.	There are 3 plates. Each plate has 2 star biscuits on. How many biscuits are there?  2 add 2 add 2 equals 6 **Use of number lines**  5 + 5 + 5 = 15 r models to show repeated addition  	Write addition sentences to describe objects and pictures. 
Arrays- showing commutative multiplication	Create arrays using counters/ cubes to show multiplication sentences.  	Draw arrays in different rotations to find **commutative** multiplication sentences.   Link arrays to area of rectangles.	Use an array to write multiplication sentences and reinforce repeated addition.  5 + 5 + 5 = 15 3 + 3 + 3 + 3 + 3 = 15 5 x 3 = 15 3 x 5 = 15

		Encourage the children to describe rectangles using appropriate language and to recognise that a 5 x 4 rectangle is the same as a 4 x 5 	
Finding out about rectangles (KS2)	Use of dienes and arrays to investigate square numbers and prime numbers **'A square number is called a square number because you can make a square with it!'** 	Using arrays to investigate numbers 'Is 25 a square number?' Why is 5 a prime number?  	What is the square root of 25? 5^2 = ?

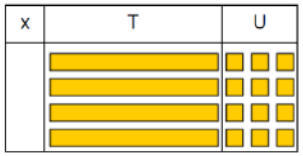
Grid Method

Show the link with arrays to first introduce the grid method.



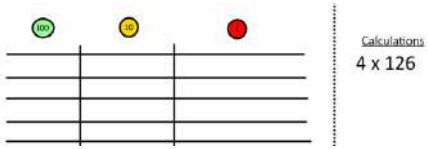
4 rows of 10
4 rows of 3

Move on to **using Base 10** to move towards a more compact method.

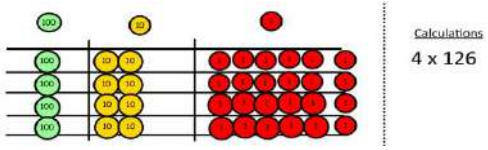


4 rows of 13 **(arrays)**

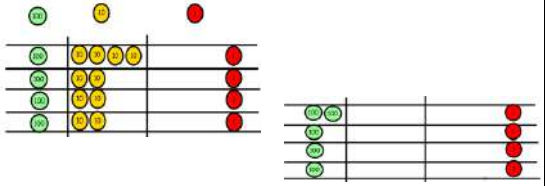
Move on to place value counters to show how we are finding groups of a number.We are multiplying by 4 so we need 4 rows.



Fill each row with 126.



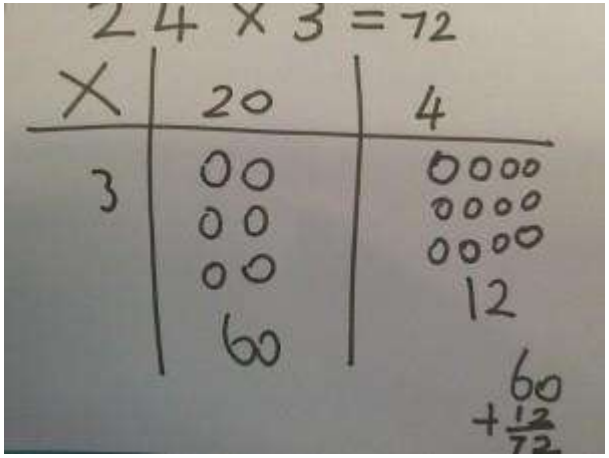
Add up each column, starting with the ones making any exchanges needed.



Then you have your answer.

Children can represent the work they have done with place value counters in a way that they understand.

They can draw the counters, using colours to show different amounts or just use circles in the different columns to show their thinking as shown below.



Start with multiplying by one digit numbers and showing the clear addition alongside the grid.

×	30	5
7	210	35

210 + 35 = 245

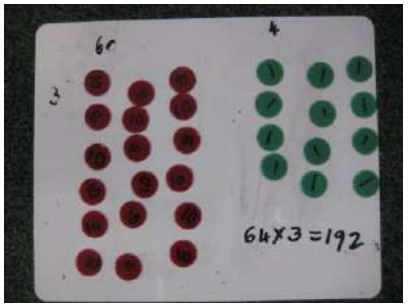
Moving forward, multiply by a 2 digit number showing the different rows within the grid method.

	10	8
10	100	80
3	30	24

X	1000	300	40	2
10	10000	3000	400	20
8	8000	2400	320	16

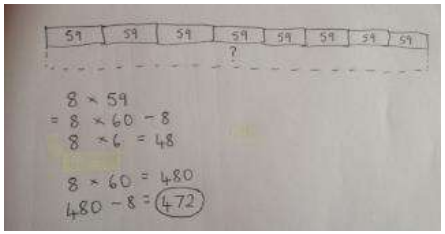
Column multiplication

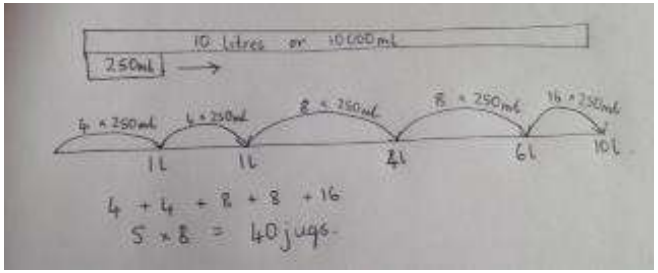
Children can continue to be supported by place value counters at the stage of multiplication.



It is important at this stage that they always multiply the ones first and note down their answer followed by the tens which they note below.

Bar modelling and number lines can support learners when solving problems with multiplication alongside the formal written methods.





This can be extended to decimal numbers.

Start with long multiplication, reminding the children about lining up their numbers clearly in columns.

If it helps, children can write out what they are solving next to their answer.

```
  32
x 24
   8   (4 x 2)
 120   (4 x 30)
  40   (20 x 2)
 600   (20 x 30)
 768
```

```
      7 4
    x 6 3
    -----
      1 2
    2 1 0
    2 4 0
+ 4 2 0 0
  ---------
  4 6 6 2
```

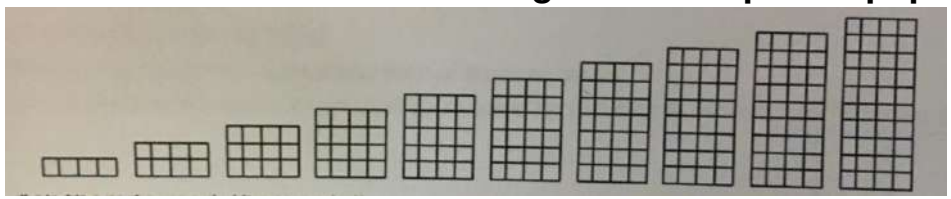
This moves to the more compact method.

```
  2 3 1
  1342
x   18
-------
 13420
 10736
-------
 24156
-------
  1
```

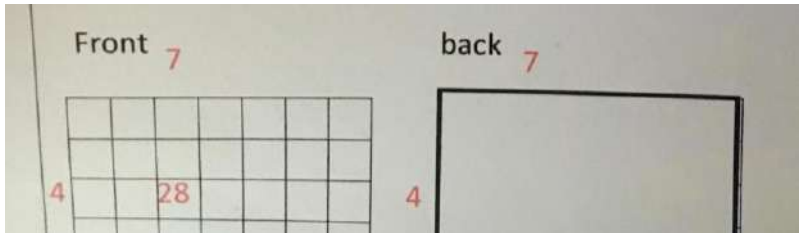
.

Learning multiplication facts

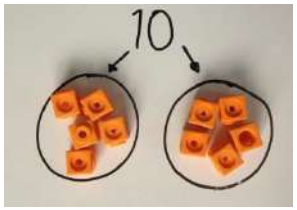
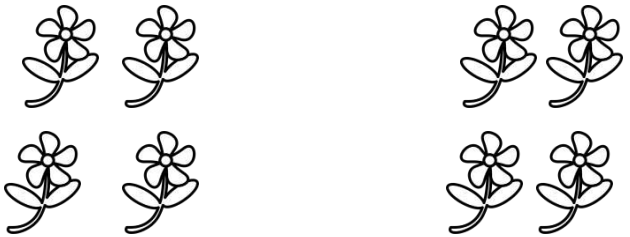
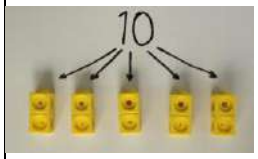
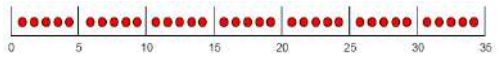
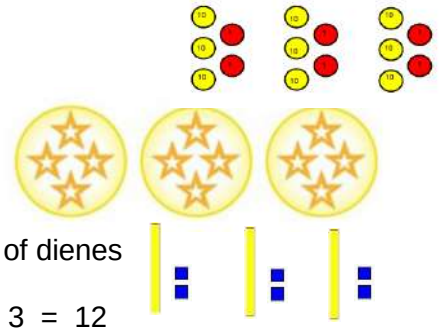
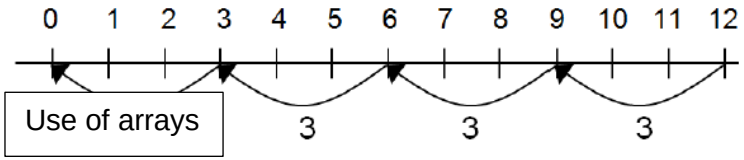
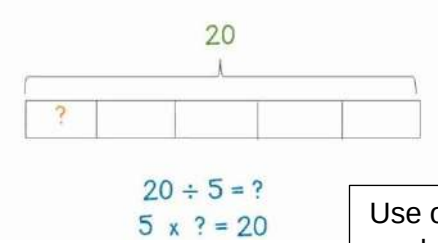
- **Ask the children to cut rectangles from squared paper to represent a set of multiplication/division facts**



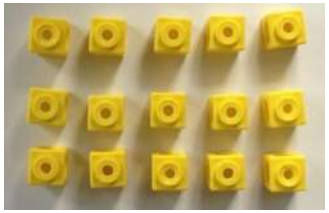
- **Write the dimensions on the back and the product on the front. Play with a partner, showing them the side without the product in the middle. Any the partner cannot work out, they can take home to learn.**



Division

Objective and Strategies	Concrete	Pictorial	Abstract
Sharing objects into groups	   I have 10 cubes, can you share them equally in 2 groups?	Children use pictures or shapes to share quantities.  8 ÷ 2 = 4 Use of dienes	Share 9 buns between three people. 9 ÷ 3 = 3
Division as grouping	Divide quantities into equal groups. Use cubes, counters, objects or place value counters to aid understanding.   96 ÷ 3 = 32  Use of dienes 36 ÷ 3 = 12	Use a n show jumps in groups. The number of jumps equals the number of groups.  Use of arrays Think of the bar as a whole. Split it into the number of groups you are dividing by and work out how many would be within each group.  Use of bar models	28 ÷ 7 = 4 Divide 28 into 7 groups. How many are in each group?

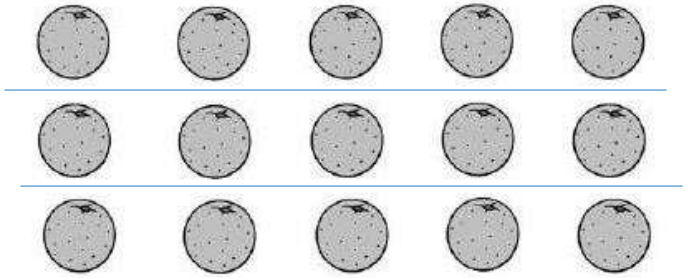
Division within arrays



Link division to multiplication by creating an array and thinking about the number sentences that can be created.

Eg 15 ÷ 3 = 5 5 x 3 = 15
15 ÷ 5 = 3 3 x 5 = 15

Use of dienes



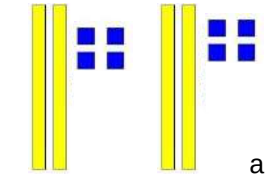
Draw an array and use lines to split the array into groups to make multiplication and division sentences.

Find the inverse of multiplication and division sentences by creating four linking number sentences.

7 x 4 = 28
4 x 7 = 28
28 ÷ 7 = 4
28 ÷ 4 = 7

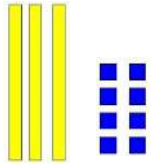
Halving (KS2)

Use of dienes

a. 

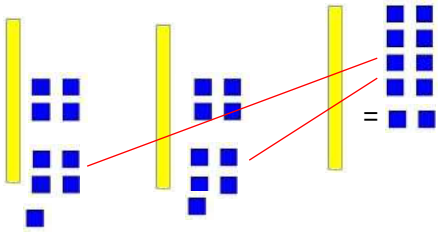
48 ÷ 2

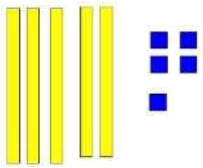
a 2-digit even number

b. 

38 ÷ 2

2-digit halving with an exchange of a ten

c. 



55 ÷ 2

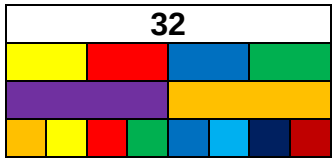
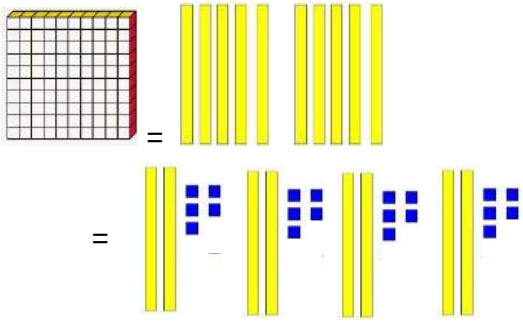
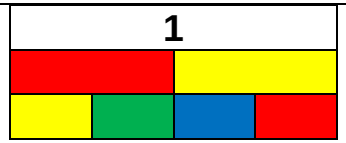
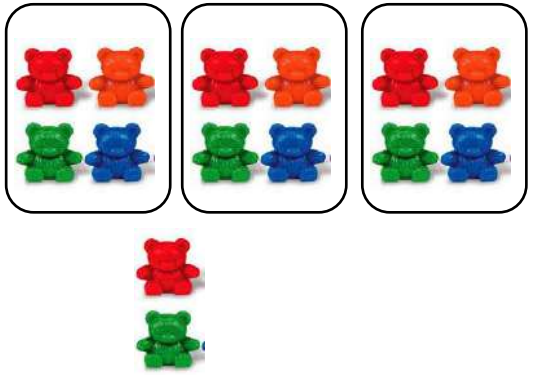
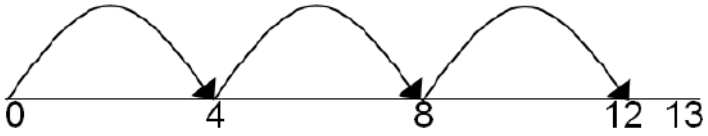
Use of bar model to represent halves

48	
24	24

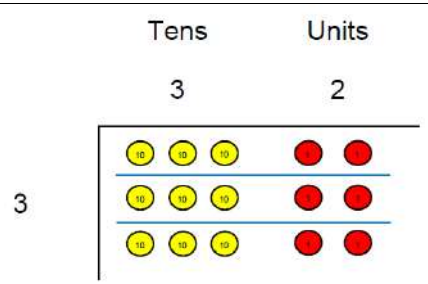
What is half of …?

½ of … = ?

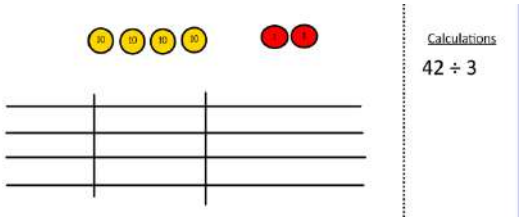
½ x 48 = ?

	2-digit halving with an exchange of a ten <u>and</u> a remainder		
Finding factors	 Use dienes to represent factors This can be extended to decimal numbers with place value counters or dienes to represent tenths and hundredths. 	 Use of bar models to investigate and represent factors	What are the factors of…? Is … a factor of …? Prove that … is a factor of… What are the common factors of … and …? What is the highest common factor of …? What are the prime factors of …?
Division with a remainder	14 ÷ 3 = Divide objects between groups and see how much is left over 	Jump forward in equal jumps on a number line then see how many more you need to jump to find a remainder.  Draw dots and group them to divide an amount and clearly show a remainder. 	Complete written divisions and show the remainder using r. 

Long division (no short division to be taught)

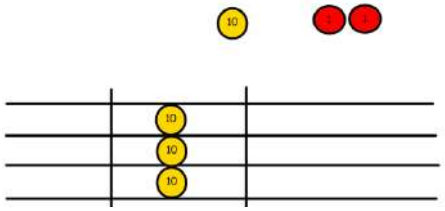


Use place value counters to divide

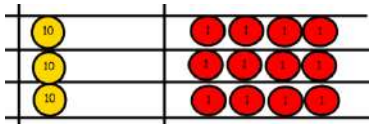


42 ÷ 3=

Start with the biggest place value, we are sharing 40 into three groups. We can put 1 ten in each group and we have 1 ten left over.



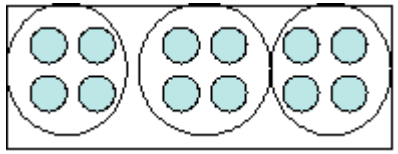
We exchange this ten for ten ones and then share the ones equally among the groups.



We look how much in 1 group so the answer is 14.

<u>Dienes can be used in place of place value counters here.</u>

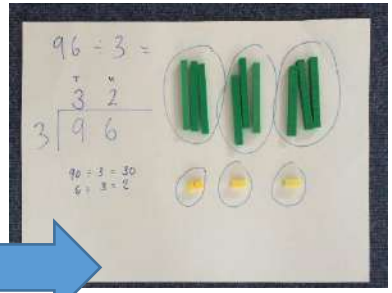
Students can continue to use drawn diagrams with dots or circles to help them divide numbers into equal groups.



Encourage them to move towards counting in multiples to divide more efficiently.

Sticks and dots can be drawn to illustrate and help calculate the division.

Dienes should be represented where exchanging will be necessary.



Begin with divisions that divide equally with no remainder. From double-digit division by a single digit onwards, the long division method must be used. Colours should be used to show how many times a number goes in, then carry out the subtraction and finally bring down the next digit.

```
   2  8
  ____
3 | 8  4
  | 6
  | ---
  | 2  4
  | 2  4
  | ---
  |    0
```

This can be extended to remainders, represented as a fraction

eg. **85 ÷ 3 = 28 r 1/3**

Then the remainder can be represented as a decimal

```
   2  8.  3
  ________
3 | 8  5.  0
  | 6
  | ---
  | 2  5
  | 2  4
  | ---
  |    1  0
  |       9
  |    ---
  |       1
```
